

A Tight Lower Bound for the Weights of Maximum Weight Matching in Bipartite Graphs

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Abstract. Let $\mathcal{G}_{m,\sigma}$ be the collection of all weighted bipartite graphs each having σ and m , as the size of a vertex partition and the total weight, respectively. We give a tight lower bound $\lceil \frac{m-\sigma}{\sigma} \rceil + 1$ for the set $\{Wt(mwm(G)) \mid G \in \mathcal{G}_{m,\sigma}\}$ which denotes the collection of weights of maximum weight bipartite matchings of all graphs in $\mathcal{G}_{m,\sigma}$.

Keywords: Maximum weight bipartite matching, Lower bound for weights of bipartite matching, Combinatorial optimization

1 Introduction

We use the notations \mathbb{N} and \mathbb{N}_0 to denote the sets of positive integers and non-negative integers, respectively. Let $G = (V = V_1 \cup V_2, E, Wt)$ be an undirected, weighted bipartite graph where V_1 and V_2 are two non-empty partitions of the vertex set V of G , and E is the edge set of G with positive integer weights on the edges which are given by the weight function $Wt: E \rightarrow \mathbb{N}$. Let W denotes the total weight of G and is defined by $W = Wt(G) = \sum_{e \in E} Wt(e)$. For uniformity we treat an unweighted graph as a weighted graph having unit weight for all edges.

We use the notation $\{u, v\}$ for an edge $e \in E$ between $u \in V_1$ and $v \in V_2$, and its weight is denoted by $Wt(e) = Wt(u, v)$. We also say that $e = \{u, v\}$ is *incident* on vertices u and v ; and u and v are each *incident* with e . Two vertices $u, v \in V$ of G are *adjacent* if there exists an edge $e = \{u, v\} \in E$ of G to which they are both incident. Two edges $e_1, e_2 \in E$ of G are *adjacent* if there exists a vertex $v \in V$ to which they are both incident [4].

A subset $M \subseteq E$ of edges is a *matching* if no two edges of M share a common vertex. A vertex $v \in V$ is said to be *covered* or *matched* by the matching M if it is incident with an edge of M ; otherwise v is *unmatched* [1, 2]. A matching M of G is called a *maximum (cardinality) matching* if there does not exist any other matching of G with greater cardinality. We denote such a matching by $mm(G)$.

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The weight of a matching M is defined as $Wt(M) = \sum_{e \in M} Wt(e)$. A matching M of G is a *maximum weight matching*, denoted as $mwm(G)$, if $Wt(M) \geq Wt(M')$ for every other matching M' of the graph G . Observe that, if G is an unweighted graph then a $mwm(G)$ is a $mm(G)$, which we write as $mwm(G) = mm(G)$ in short and its weight is given by $Wt(mwm(G)) = |mm(G)|$. Similarly, if G is an undirected and weighted graph with $Wt(e) = c$ for all edges e in G and c is a constant then also we have $mwm(G) = mm(G)$ with weight of the matching as $Wt(mwm(G)) = c * |mm(G)|$.

Maximum Weight Bipartite Matching (MWBM) problem is a well studied problem in combinatorial optimization and algorithmics, and has wide range of applications (see textbooks [13, 15]). Several exact, approximate and randomized algorithms have also been proposed for computing maximum weight bipartite matching [5–9, 11, 12, 14].

Our Contribution. In this paper we give a tight lower bound for the weights of MWBM in bipartite graphs having fixed weight and vertex size. Let $\mathcal{G}_{m,\sigma}$ be the collection of all weighted bipartite graphs, each of whose weight is m and σ is the size of each partition of the vertex set, where m and σ are positive integers. The set of weights of MWBM of the graphs in $\mathcal{G}_{m,\sigma}$ is denoted by $\{Wt(mwm(G)) \mid G \in \mathcal{G}_{m,\sigma}\}$. We prove that $\lceil \frac{m-\sigma}{\sigma} \rceil + 1$ is a lower bound of $\{Wt(mwm(G)) \mid G \in \mathcal{G}_{m,\sigma}\}$ and this bound is tight. This result is useful in telecommunication network (with minimum cost, minimum time, minimum traffic load analysis, critical path routing), monitoring computer network, computer vision, pattern recognition, machine learning [3], stringology [10], and also in compiler design with cloud architecture.

Roadmap. The rest of the paper is organized as follows. In Section 2 we partition the class of graphs in $\mathcal{G}_{m,\sigma}$ into two subclasses and provide a tight lower bound for the weights of MWBM of graphs in $\mathcal{G}_{m,\sigma}$. A summary is given in Section 3.

2 A Tight Lower Bound for the Weights of Maximum Weight Bipartite Matching in $\mathcal{G}_{m,\sigma}$

Let $\mathcal{G}_{m,\sigma}$ denotes the collection of all weighted bipartite graphs, each of whose weight is fixed to m and σ is the size of each of the two vertex partitions of any graph in $\mathcal{G}_{m,\sigma}$. Let us consider the pair of non-empty partitions of the vertex sets of all the bipartite graphs in $\mathcal{G}_{m,\sigma}$ be Σ_P and Σ_T . Therefore, $|\Sigma_P| = |\Sigma_T| = \sigma$. We partition $\mathcal{G}_{m,\sigma}$ as $\mathcal{G}_{m,\sigma} = \mathcal{G}_{m \geq \sigma} \cup \mathcal{G}_{m < \sigma}$, where

$$\mathcal{G}_{m \geq \sigma} \equiv \{G = (\Sigma_P \cup \Sigma_T, E, Wt) \mid \sigma = |\Sigma_P| = |\Sigma_T|, m = Wt(G), m \geq \sigma\}$$

and

$$\mathcal{G}_{m < \sigma} \equiv \{G = (\Sigma_P \cup \Sigma_T, E, Wt) \mid \sigma = |\Sigma_P| = |\Sigma_T|, m = Wt(G), m < \sigma\}.$$

Now we prove that the value of $\min_{G \in \mathcal{G}_{m,\sigma}} \{Wt(mwm(G))\}$, which denotes minimum weight among the maximum weight bipartite matchings of all the graphs in $\mathcal{G}_{m,\sigma}$, is $\lceil \frac{m-\sigma}{\sigma} \rceil + 1$. Let us first prove it for $\mathcal{G}_{m \geq \sigma} \subseteq \mathcal{G}_{m,\sigma}$.

Since $m \geq \sigma$, we can always write m as $q\sigma + r$ for some $q, r \in \mathbb{N}_0$ where $0 < r \leq \sigma$. First we show the existence of bipartite graph $G \in \mathcal{G}_{m \geq \sigma}$ such that $Wt(mwm(G)) = q + 1$. We then prove in Theorem 2 that $q + 1$ is the tight lower bound of the set $\{Wt(mwm(G)) \mid G \in \mathcal{G}_{m \geq \sigma}\}$.

Theorem 1 (Existence of Bipartite Graph in $\mathcal{G}_{m \geq \sigma}$ with Weight of the MWBM Equal to $q + 1$). *Let $\mathcal{G}_{m \geq \sigma} = \{G = (\Sigma_P \cup \Sigma_T, E, Wt) \mid \sigma = |\Sigma_P| = |\Sigma_T|, m = Wt(G) \text{ and } m \geq \sigma\}$. If $m = q\sigma + r$ for some non-negative integers q and r where $0 < r \leq \sigma$, then there exists a bipartite graph $G \in \mathcal{G}_{m \geq \sigma}$ such that $Wt(mwm(G)) = q + 1$.*

Proof. For the case $q = 0$, we have $m = q\sigma + r = r = \sigma$ as $0 < r \leq \sigma$ and $m \geq \sigma$. Figure 1(a) shows a bipartite graph $G' = (\Sigma_P \cup \Sigma_T, E', Wt) \in \mathcal{G}_{m \geq \sigma}$ for this case. The weight of the graph is $Wt(G') = \sigma$. In this graph G' , $Wt(mwm(G')) = 1 = q + 1$.

For $q \geq 1$, the total weight of any bipartite graph in $\mathcal{G}_{m \geq \sigma}$ is $m = q\sigma + r$. We produce such a bipartite graph $G'' \in \mathcal{G}_{m \geq \sigma}$ shown in Figure 1(b) with $Wt(mwm(G'')) = q + 1$. \square

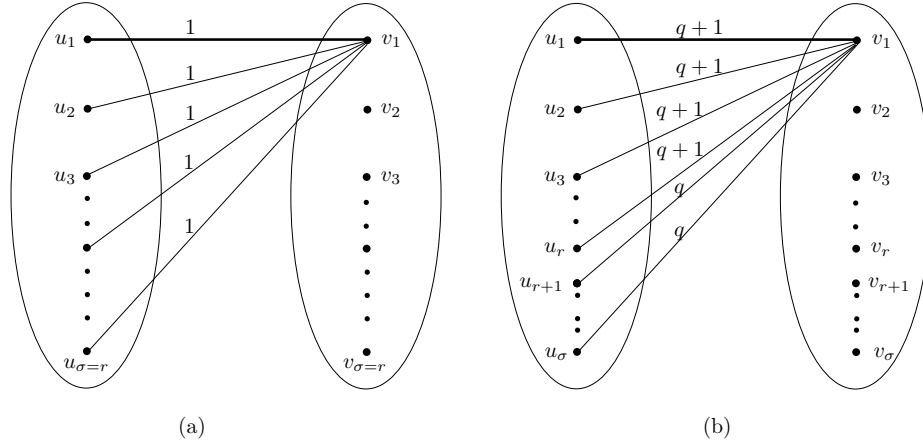


Fig. 1: Given $m = q\sigma + r$ for some $q, r \in \mathbb{N}_0$ where $0 < r \leq \sigma$ and $\mathcal{G}_{m \geq \sigma} = \{G = (\Sigma_P \cup \Sigma_T, E, Wt) \mid \sigma = |\Sigma_P| = |\Sigma_T|, m = Wt(G), m \geq \sigma\}$ such that $\min_{G \in \mathcal{G}_{m \geq \sigma}} \{Wt(mwm(G))\} = q + 1$. (a) An example of bipartite graph for the case $q = 0$. (b) An example of bipartite graph for the case $q \geq 1$. In both the graphs the thick edge represents maximum weight matching edge.

Observe that in a weighted graph G , any edge e of weight $c \in \mathbb{N}$ can be thought of as c number of overlapping unit weight edges. Similarly, increasing the weight of a bipartite graph G by adding a weight $c \in \mathbb{N}$ is equivalent to adding c unit

weight edges in G . Without loss of generality, we assume these as a convention while incrementing weight in a weighted graph.

Theorem 2 (Tight Lower Bound for the Weights of MWBM of the Graphs in $\mathcal{G}_{m \geq \sigma}$). *Let $\mathcal{G}_{m \geq \sigma} = \{G = (\Sigma_P \cup \Sigma_T, E, Wt) \mid \sigma = |\Sigma_P| = |\Sigma_T|, m = Wt(G) \text{ and } m \geq \sigma\}$. Then*

$$\min_{G \in \mathcal{G}_{m \geq \sigma}} \{Wt(mwm(G))\} = q + 1$$

where $m = q\sigma + r$ for some non-negative integers q and r , and $0 < r \leq \sigma$.

Proof. For $\sigma = 1$, the statement is trivially true. So we consider $\sigma \geq 2$ and prove the statement $\min_{G \in \mathcal{G}_{m \geq \sigma}} \{Wt(mwm(G))\} = q + 1$ by induction on $q \in \mathbb{N}_0$. Let $\Sigma_P = \{u_1, u_2, \dots, u_\sigma\}$ and $\Sigma_T = \{v_1, v_2, \dots, v_\sigma\}$ be the disjoint vertex sets of the graphs in $\mathcal{G}_{m \geq \sigma}$. For simplicity, we denote $\mathcal{G}_{q+1} = \mathcal{G}_{m \geq \sigma}$ when $m = q\sigma + r$ for some $q, r \in \mathbb{N}_0$ where $0 < r \leq \sigma$, that is, $q = \lceil \frac{m-\sigma}{\sigma} \rceil$ where q is represented as a function of m and σ only.

Base Step: Let $q = 0$. Then $m = r = \sigma$ because $0 < r \leq \sigma$ and $m \geq \sigma$, and

$$\mathcal{G}_1 = \{G = (\Sigma_P \cup \Sigma_T, E, Wt) \mid \sigma = |\Sigma_P| = |\Sigma_T|, Wt(G) = \sigma\}.$$

Since for any graph $G = (\Sigma_P \cup \Sigma_T, E, Wt) \in \mathcal{G}_1$, $|\Sigma_P| = |\Sigma_T| = \sigma$ and $Wt(G) = \sigma$, therefore $\min_{G \in \mathcal{G}_1} \{Wt(mwm(G))\} = 1 = q + 1$.

Induction Hypothesis: Assume that for $q = i$, $\min_{G \in \mathcal{G}_{i+1}} \{Wt(mwm(G))\} = i + 1$, where

$$m = i\sigma + r, \text{ and}$$

$$\mathcal{G}_{i+1} = \{G = (\Sigma_P \cup \Sigma_T, E, Wt) \mid \sigma = |\Sigma_P| = |\Sigma_T|, Wt(G) = i\sigma + r\}.$$

Let $\mathcal{G}'_{i+1} = \{G \in \mathcal{G}_{i+1} \mid Wt(mwm(G)) = i + 1\}$. The set \mathcal{G}'_{i+1} is non-empty by the Theorem 1. We use this set in the following inductive step.

Inductive Step: Let $q = i + 1$. We have to prove that $\min_{G \in \mathcal{G}_{i+2}} \{Wt(mwm(G))\} = i + 2$, where

$$m = (i + 1)\sigma + r, \text{ and}$$

$$\mathcal{G}_{i+2} = \{G = (\Sigma_P \cup \Sigma_T, E, Wt) \mid \sigma = |\Sigma_P| = |\Sigma_T|, Wt(G) = (i + 1)\sigma + r\}.$$

The existence of a graph $G \in \mathcal{G}_{i+2}$ with $Wt(mwm(G)) = i + 2$ is proved in Theorem 1. Therefore, we only have to prove that there does not exist any graph in \mathcal{G}_{i+2} whose weight of a maximum weight matching is $i + 1$. Let us prove it by contradiction. Suppose there exists a graph $G_* \in \mathcal{G}_{i+2}$ such that $Wt(mwm(G_*)) = i + 1$.

Observe that, for any graph in \mathcal{G}_{i+2} , its weight is equal to $m = (i + 1)\sigma + r = (i\sigma + r) + \sigma$. Therefore, any graph in \mathcal{G}_{i+2} is generated by adding a total of σ weight to the non-negative weight edges of a graph in \mathcal{G}_{i+1} .

Therefore, G_* can only be constructed from a graph in \mathcal{G}'_{i+1} by adding a total of σ weight to the non-negative weight edges of that graph in \mathcal{G}'_{i+1} ; because for all $G \in \mathcal{G}_{i+1} \setminus \mathcal{G}'_{i+1}$, $Wt(mwm(G)) > i + 1$. Let $\Sigma = \{e_1, e_2, e_3, \dots\}$ be the edges, where $\sigma = \sum_{e_i \in \Sigma} Wt(e_i)$, whose weights are increased in $G \in \mathcal{G}'_{i+1}$ to build G_* .

Case 1. Let $G \in \mathcal{G}'_{i+1}$ and $M = mwm(G)$. If there exists at least one edge e in Σ such that $e \in M$ or if both the end points of e are unmatched vertices, then let $M' = M \cup \{e\}$, which is a weighted matching of G_* , not necessarily of maximum weight. Therefore

$$Wt(mwm(G_*)) \geq Wt(M') = Wt(M) + Wt(e) = i + 1 + Wt(e) > i + 1$$

which is a contradiction because we assumed that $(mwm(G_*)) = i + 1$.

Note: Hence for the rest of the cases we assume that none of the edges in Σ , which are added in $G \in \mathcal{G}'_{i+1}$ to get the $G_* \in \mathcal{G}_{i+2}$, belongs to M ; or both the end points of none of the edges in Σ are unmatched vertices. Therefore if $e = \{u, v\} \in \Sigma$, then: (a) u is an unmatched vertex and v is a matched vertex or vice versa, or (b) both u and v are matched vertices, but $e \notin M = mwm(G)$.

Case 2. Let there exists at least one edge $e = \{u, v\} \in \Sigma$ such that $Wt(e) = w_\sigma \geq 2$. Then we have the following two sub-cases which are shown in Figure 2. Let $G \in \mathcal{G}'_{i+1}$ and $M = mwm(G)$.

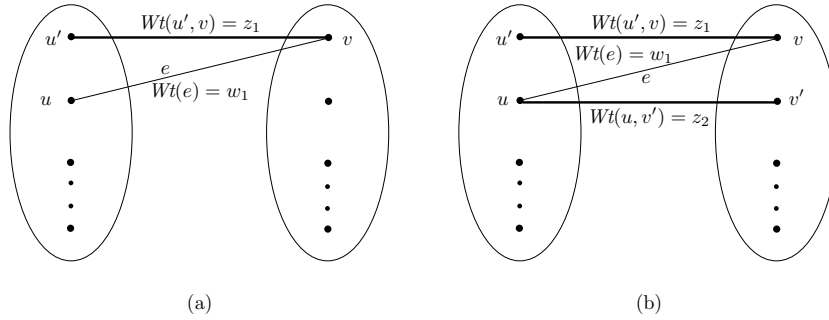


Fig. 2: **(a)** This graph gives a pictorial representation of the Sub-case 2(a) in Theorem 2. **(b)** Sketch of the graph considered in Sub-case 2(b) is shown here. In both the graphs the thick edges are maximum weight matching edges.

Sub-case 2(a): Assume that u and v be the unmatched and matched vertices in $G \in \mathcal{G}'_{i+1}$, respectively. So there exists an edge $e' = \{u', v\} \in M$ which is incident on the matched vertex v . Let $Wt(e') = Wt(u', v) = z_1$ and $Wt(e) = Wt(u, v) = w_1$ in the G . Therefore $z_1 \geq w_1$. Now add the edge e (or increase the edge weight of e) in G where $Wt(e) = w_\sigma \geq 2$ in order to generate $G_* \in \mathcal{G}_{i+2}$ such that $Wt(mwm(G_*)) = i + 1$. If $z_1 < w_1 + w_\sigma$, then let

$$M' = M \setminus \{e'\} \cup \{e\}$$

which is a weighted matching of G_* . Hence

$$\begin{aligned} \text{Wt}(\text{mwm}(G_*)) &\geq \text{Wt}(M') = \text{Wt}(M) - z_1 + w_1 + w_\sigma \\ &= (i+1) - z_1 + w_1 + w_\sigma \\ &> i+1 \end{aligned}$$

which is a contradiction.

Or else,

$$z_1 \geq w_1 + w_\sigma \Leftrightarrow z_1 - 1 \geq w_1 + (w_\sigma - 1).$$

Therefore we can construct a new graph G' from G by decreasing one unit weight of the edge $e' = \{u', v\} \in M$ and increasing the weight of the edge $e = \{u, v\} \notin M$ by one unit in G . As a consequence, the weight of G' remains the same as that of $G \in \mathcal{G}'_{i+1}$ and so $G' \in \mathcal{G}_{i+1}$. But

$$\text{Wt}(\text{mwm}(G')) = i < \text{Wt}(M) = i+1$$

which contradicts the induction hypothesis that $\min_{G \in \mathcal{G}_{i+1}} \{\text{Wt}(\text{mwm}(G))\} = i+1$.

Sub-case 2(b): Suppose both u and v are matched vertices but $e = \{u, v\} \notin M$. See Figure 2(b). So there exist two edges $e' = \{u', v\} \in M$ and $e'' = \{u, v'\} \in M$ which are incident on the matched vertices v and u , respectively. Let $\text{Wt}(e') = z_1$, $\text{Wt}(e'') = z_2$ and $\text{Wt}(e) = w_1$ in $G \in \mathcal{G}'_{i+1}$.

$$\therefore z_1 + z_2 \geq w_1 \quad \text{in } G.$$

Now after adding the edge e in G with $\text{Wt}(e) = w_\sigma \geq 2$, if

$$z_1 + z_2 < w_1 + w_\sigma,$$

then let

$$M' = M \setminus \{e', e''\} \cup \{e\}$$

which is a weighted matching of G_* . Hence

$$\begin{aligned} \text{Wt}(\text{mwm}(G_*)) &\geq \text{Wt}(M') = \text{Wt}(M) - z_1 - z_2 + w_1 + w_\sigma \\ &= (i+1) - z_1 - z_2 + w_1 + w_\sigma \\ &> i+1 \end{aligned}$$

which is a contradiction.

Or else,

$$z_1 + z_2 \geq w_1 + w_\sigma \Leftrightarrow (z_1 - 1) + z_2 \geq w_1 + (w_\sigma - 1).$$

Therefore we can construct a new graph G' from G by reducing one unit weight of the edge $e' = \{u', v\} \in M$ and adding one unit weight to the edge $e = \{u, v\} \notin M$ of G . As a consequence, the weight of G' is the same as that of $G \in \mathcal{G}'_{i+1}$ and so $G' \in \mathcal{G}_{i+1}$. But

$$\text{Wt}(\text{mwm}(G')) = i < \text{Wt}(M) = i+1$$

which contradicts the hypothesis that $\min_{G \in \mathcal{G}_{i+1}} \{\text{Wt}(\text{mwm}(G))\} = i+1$.

Case 3. Let for each edge $e \in \Sigma$, $Wt(e) = 1$. Consider $\Sigma = \{e_1 = \{u_1, v_1\}, e_2 = \{u_2, v_2\}, \dots, e_\sigma = \{u_\sigma, v_\sigma\}\}$ and their respective weights in $G \in \mathcal{G}'_{i+1}$ are given by $\{w_1, w_2, \dots, w_\sigma\}$. We add these σ number of edges of Σ in $G \in \mathcal{G}'_{i+1}$ to produce a graph $G_* \in \mathcal{G}_{i+2}$ such that $Wt(mwm(G_*)) = i + 1$. Further let $M = mwm(G)$.

Therefore, there must exist two edges in Σ which are not adjacent. Because if not, then all the edges of Σ are adjacent to one vertex. Without loss of generality, suppose $u_1 = u_2 = \dots = u_\sigma$. See Figure 3 and consider the following two possibilities.

- (a) If $u_1 \in \Sigma_P$ is an unmatched vertex in $G \in \mathcal{G}'_{i+1}$, then there must be another unmatched vertex in Σ_T of the graph G , because $\sigma = |\Sigma_P| = |\Sigma_T|$. Say the unmatched vertex is $v_1 \in \Sigma_T$. If we add an edge $\{u_1, v_1\} \in \Sigma$ in $G \in \mathcal{G}'_{i+1}$, then this kind of graph is already addressed in Case 1. Therefore, at most $\sigma - 1$ number of edges of unit weight can be added in G while generating the G_* . This is a contradiction.
- (b) Similarly, if $u_1 \in \Sigma_P$ is a matched vertex in $G \in \mathcal{G}'_{i+1}$, then there must be another matched vertex in Σ_T of the graph G . The rest of the argument is similar to previous unmatched case.

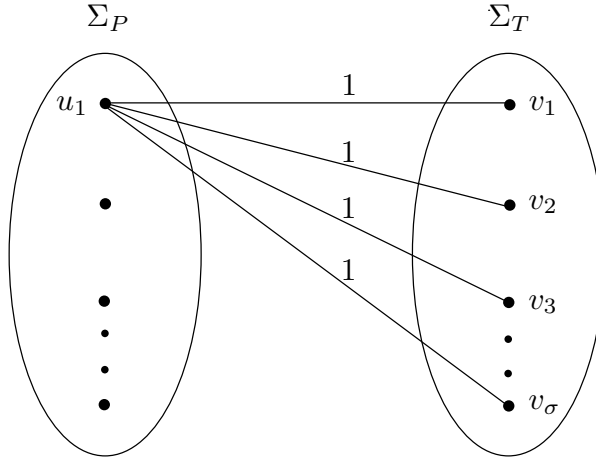


Fig.3: There must exists two edges $e_1, e_2 \in \Sigma$ such that e_1 and e_2 are not adjacent. This kind of graph does not arise in Case 3 of Theorem 2.

So we assume the two non-adjacent edges be $e_1, e_2 \in \Sigma$. Then a maximum of four edges in M are adjacent to edges $e_1, e_2 \in \Sigma$. Let e'_1, e'_2, e'_3, e'_4 be such edges and z_1, z_2, z_3, z_4 be their corresponding weights in $G \in \mathcal{G}'_{i+1}$, respectively.

$$\therefore z_1 + z_2 + z_3 + z_4 \geq w_1 + w_2 \quad \text{in } G.$$

Now after adding σ edges of Σ in G , if

$$z_1 + z_2 + z_3 + z_4 < w_1 + w_2 + 2,$$

then let

$$M' = M \setminus \{e'_1, e'_2, e'_3, e'_4\} \cup \{e_1, e_2\}$$

which is a weighted matching of G_* . Hence

$$\begin{aligned} \text{Wt}(mwm(G_*)) &\geq \text{Wt}(M') \\ &= \text{Wt}(M) - (z_1 + z_2 + z_3 + z_4) + (w_1 + w_2 + 2) \\ &> i + 1 \end{aligned}$$

which is a contradiction.

Or else,

$$\begin{aligned} z_1 + z_2 + z_3 + z_4 &\geq w_1 + w_2 + 2 \\ \Leftrightarrow z_1 + z_2 + z_3 + z_4 - 1 &\geq w_1 + w_2 + 1. \end{aligned}$$

As a consequence, by similar argument as stated in Sub-case 2(b), we can construct a new graph G' whose weight is same as that of $G \in \mathcal{G}'_{i+1}$ and so $G' \in \mathcal{G}_{i+1}$. But

$$\text{Wt}(mwm(G')) = i < \text{Wt}(M) = i + 1$$

which contradicts the induction hypothesis that $\min_{G \in \mathcal{G}_{i+1}} \{\text{Wt}(mwm(G))\} = i + 1$.

This completes the proof. \square

An equivalent statement of the Theorem 2 is the following.

Corollary 1. *For the partition $\mathcal{G}_{m \geq \sigma} \equiv \{G = (\Sigma_P \cup \Sigma_T, E, \text{Wt}) \mid \sigma = |\Sigma_P| = |\Sigma_T|, m = \text{Wt}(G) \text{ and } m \geq \sigma\}$*

$$\min_{G \in \mathcal{G}_{m \geq \sigma}} \{\text{Wt}(mwm(G))\} = \left\lceil \frac{m - \sigma}{\sigma} \right\rceil + 1.$$

Proof. Since $m \geq \sigma$, we can always write m as $q\sigma + r$ for some $q, r \in \mathbb{N}_0$ where $0 < r \leq \sigma$. Then the term $\left\lceil \frac{m - \sigma}{\sigma} \right\rceil$ can be written as

$$\left\lceil \frac{m - \sigma}{\sigma} \right\rceil = \left\lceil \frac{q\sigma + r - \sigma}{\sigma} \right\rceil = \left\lceil \frac{(q - 1)\sigma + r}{\sigma} \right\rceil = (q - 1) + 1 = q.$$

Hence the statement in this corollary is equivalent to Theorem 2. \square

The following theorem is for the partition of graphs in $\mathcal{G}_{m < \sigma}$. The proof is trivial. Note that for $0 < m < \sigma$, the term $\left\lceil \frac{m - \sigma}{\sigma} \right\rceil + 1 = 1$.

Theorem 3. *For the partition $\mathcal{G}_{m < \sigma} \equiv \{G = (\Sigma_P \cup \Sigma_T, E, \text{Wt}) \mid \sigma = |\Sigma_P| = |\Sigma_T|, m = \text{Wt}(G) \text{ and } m < \sigma\}$*

$$\min_{G \in \mathcal{G}_{m < \sigma}} \{\text{Wt}(mwm(G))\} = 1.$$

3 Conclusion

Thus we have given a tight lower bound $\lceil \frac{m-\sigma}{\sigma} \rceil + 1$ for the weights of maximum weight matching of bipartite graphs each having fixed weight as m and size of a vertex partition as σ .

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